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A note on modulated cross-waves

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Larraza & Putterman (1984) and Miles (1984*b*) derived a nonlinear Schrödinger (NLS) equation for the envelope $r(X, \tau)$ of a spatially and temporally modulated cross-wave for which the spatial mean square $\overline{|r|^2}$ vanishes (e.g. a solitary wave). Sasaki (1993) found that the accommodation of non-vanishing $\overline{|r|^2}$ (e.g. a cnoidal wave) introduces a non-local term proportional to $\overline{|r|^2}$ in the coefficient of r in the NLS equation. Sasaki's result is confirmed through an average-Lagrangian formulation, in which the functional $\overline{|r|^2}$ appears (after appropriate normalization) as the Lagrange multiplier associated with the constraint of conservation of mass for the envelope. This functional is a constant, and implies a quadratic (in amplitude) shift of the resonant frequency, for a periodic ($\partial r/\partial\tau = 0$) wave; but if the X and τ dependencies of r are not separable it implies the replacement of the NLS equation by a nonlinear integral-partial-differential equation.

1. Introduction

Sasaki (1993) has remarked that the generalized nonlinear Schrödinger (NLS) equation (Larraza & Putterman 1984; Miles 1984*b*)

$$\mathrm{i}(r_\tau + \alpha r) + B r_{XX} + (\beta + A|r|^2)\, r + \gamma r^* = 0 \tag{1.1}$$

for the complex amplitude $r(X, \tau)$ (X and τ are slow space and time variables, and the parameters α, β, A and B are defined in §2 below) of the envelope of a parametrically excited, modulated cross-wave implicitly assumes that the spatial mean square $\overline{|r|^2}$ vanishes (as is true for the solitary waves considered by Larraza & Putterman and Miles) and therefore is invalid for the cnoidal-wave solutions in Appendix A of Miles (1984*b*) and in Umeki (1991) or for the kink solutions of Denardo *et al.* (1990) and (although Sasaki does not give this reference) Guthart & Wu (1991). (Pierce & Knobloch 1994 find related, non-local effects in the evolution equations for edge waves.) Sasaki finds that the coefficient of r in (1.1) comprises an additional term proportional to $\overline{|r|^2}$. If $r_\tau = 0$, as for a periodic wave with a stationary envelope, $\overline{|r|^2}$ is a constant that may be incorporated in β and implies a shift in the resonant frequency.† The NLS form also is retained if the X and τ dependencies of r are separable, but if r is both non-local and non-separable (as in the stability analysis of a cnoidal envelope) the NLS equation is replaced by a nonlinear, integral-partial-differential equation.

Sasaki's result for non-local waves appears to be of sufficient interest to warrant an independent derivation through the average-Lagrangian formulation of Miles (1984*a*, *b*, hereinafter referred to as M84*a*, *b*). I sketch such a derivation in §2 and find

† This result is adumbrated by Larraza & Putterman's (1984) equation (24), but that result is developed in the context of compact support.

that $\overline{|r|^2}$ is (after appropriate normalization) the Lagrange multiplier associated with the conservation-of-mass constraint for the envelope. I then go on, in §3, to consider periodic waves and correct the cnoidal-wave solution in Appendix A of M84*b*.

2. Evolution equations

We consider weakly modulated cross-waves induced in a rectangular channel of breadth b, depth d and length $l \gg b$ that is subjected to the vertical oscillation

$$z_0 = a_0 \cos 2\omega t \quad (0 < \omega^2 a_0/g \ll 1), \tag{2.1}$$

where ω approximates the natural frequency (for gravity waves, but capillarity is significant in typical experiments and may be incorporated as in M84*a, b* and Miles 1985)

$$\omega_1 = (gk \tanh kd)^{1/2} \quad (k = \pi/b) \tag{2.2}$$

of the dominant cross-wave. The velocity potential at the free surface and the displacement of that surface (relative to the plane of the level surface), ξ and η, admit the Fourier expansions

$$(\xi, \eta) = \sum_{n=0}^{\infty} (\xi_n, \eta_n)\, \psi_n(y), \quad \psi_n = (2-\delta_{0n})^{1/2} \cos nky \quad (0 < y < b), \tag{2.3a, b}$$

where ξ_n and η_n are canonical variables and δ_{0n} is the Kronecker delta. The assumptions

$$ka \equiv 2\epsilon^{1/2} \tanh kd, \quad \beta \equiv \frac{\omega^2 - \omega_1^2}{2\epsilon\omega_1^2} = O(1), \quad \gamma \equiv \frac{\omega^2 a_0}{\epsilon g} = O(1), \tag{2.4a–c}$$

where a is an amplitude scale and ϵ is a small parameter, permit the η_n to be posed in the form (only $n = 0, 1, 2$ are significant in the present approximation)

$$\eta_n = \delta_{1n}\, a(p \cos\theta + q \sin\theta) + a^2 k \tanh kd\,(A_n \cos 2\theta + B_n \sin 2\theta + C_n), \tag{2.5}$$

where $\theta \equiv \omega t$ and p, q, A_n, B_n and C_n are functions of the slow variables

$$\tau \equiv \epsilon\omega t, \quad X \equiv 2(\epsilon \tanh kd)^{1/2} kx. \tag{2.6a, b}$$

$A_1 = B_1 = C_1 = 0$, and A_2, B_2 and C_2 are determined as quadratic functions of p and q in M84*a*, §3. The evolution equations for p and q are derived in M84*b* on the implicit assumption that $\iint \eta^2 \,\mathrm{d}x\,\mathrm{d}y = 0$ (in which sense η is *local*). We now posit the weaker constraint

$$\mathscr{M} \equiv \iint \eta \,\mathrm{d}x\,\mathrm{d}y = b \int \eta_0 \,\mathrm{d}x = 0, \tag{2.7}$$

where, here and subsequently, the limits of integration for integrals with respect to x and y are $(0, l)$ and $(0, b)$, respectively. The corresponding Lagrangian is

$$\mathscr{L} = \iint L \,\mathrm{d}x\,\mathrm{d}y, \quad L = \xi\eta_t - \tfrac{1}{2}\int_{-d}^{\eta} (\nabla\phi)^2 \,\mathrm{d}z - \tfrac{1}{2}(g + \ddot{z}_0)\,\eta^2, \tag{2.8a, b}$$

where ϕ is the velocity potential. The evolution equations for the ξ_n and η_n are determined by $\delta\mathscr{L} = 0$ (Hamilton's principle) subject to the constraint (2.7) or, equivalently, by

$$\delta(\mathscr{L} + \lambda\mathscr{M}) = \delta \iint (L + \lambda\eta) \,\mathrm{d}x\,\mathrm{d}y = 0, \tag{2.9}$$

where λ is a Lagrange multiplier (to be determined).

Proceeding as in M84*b*, §3, we invoke (2.9) for the variations $\delta\xi_0$ and $\delta\eta_0$ to obtain

$$\eta_{0t} = 0, \quad \xi_{0t} = \lambda - g\eta_0 - \tfrac{1}{4}(T^{-2}-1)\,\eta_{1t}^2, \quad T \equiv \tanh kd, \qquad (2.10\,a\text{–}c)$$

which differ from M84*b* (3.11) only in the presence of λ. Taking the x and θ averages, which we denote by $\overline{}$ and $\langle\ \rangle$, respectively, of (2.10*b*), invoking $\overline{\eta_0} = 0$, which follows from (2.7), and $\langle \xi_{0t}\rangle = 0$, which follows from the requirement that $\nabla\xi_0$ be bounded as $t\uparrow\infty$ (see M84*b*, §3), and introducing $r \equiv p + iq$ and $r^* \equiv p - iq$, we obtain

$$\lambda = C\overline{\langle\eta_{1t}^2\rangle} = \tfrac{1}{2}C\omega^2a^2\overline{|r|^2}, \quad C \equiv \tfrac{1}{4}(T^{-2}-1), \qquad (2.11\,a,b)$$

$$g\eta_0 = \tfrac{1}{2}C\omega^2a^2(\overline{|r|^2} - |r|^2), \quad \xi_{0t} = -\tfrac{1}{4}C\omega^2a^2(r^2\,\mathrm{e}^{-2\mathrm{i}\theta} + r^{*2}\,\mathrm{e}^{2\mathrm{i}\theta}). \qquad (2.12\,a,b)$$

The corresponding approximation to $\langle L + \lambda\eta\rangle$, calculated as in M84*a*, *b*, is, after eliminating λ, η_0 and ξ_0 through (2.11) and (2.12),

$$\langle L+\lambda\eta\rangle = \tfrac{1}{2}\epsilon g a^2 [\tfrac{1}{2}\mathrm{i}(r_\tau r^* - r r_\tau^*) - B|r_X|^2 + \beta|r|^2 + \tfrac{1}{2}\gamma(r^2 + r^{*2}) + \tfrac{1}{2}A_0|r|^4 + \tfrac{1}{2}(A - A_0)(|r|^2 - \overline{|r|^2})^2], \quad (2.13)$$

where

$$A_0 \equiv \tfrac{1}{8}(2T^4 + 3T^2 + 12 - 9T^{-2}), \quad B \equiv T + kd(1 - T^2), \qquad (2.14\,a,b)$$

and

$$A \equiv A_0 + \tfrac{1}{2}(1-T^2)^2 = \tfrac{1}{8}(6T^4 - 5T^2 + 16 - 9T^{-2}). \qquad (2.14\,c)$$

Invoking $\delta\langle \mathscr{L} + \lambda\mathscr{M}\rangle/\delta r^* = 0$ (see the Appendix regarding the variation of $\int[|r|^2 - \overline{|r|^2}]^2\,\mathrm{d}x$) and introducing linear damping, we obtain

$$\mathrm{i}(r_\tau + \alpha r) + Br_{XX} + [\beta + A|r|^2 - (A - A_0)\overline{|r|^2}]\,r + \gamma r^* = 0, \qquad (2.15)$$

where $\alpha = \delta/\epsilon$ and δ is the ratio of actual to critical damping for free oscillations. Equation (2.15) is equivalent to Sasaki's (1993) (4.3) and reduces to (1.1) above for $\overline{|r|^2} = 0$. Note that $A - A_0$, the coefficient of $\overline{|r|^2}$ in (2.15), vanishes for $kd \gg 1$.

3. Periodic solutions

We now consider periodic (in t) solutions, for which $\partial_\tau = 0$, $\overline{|r|^2}$ is a constant, and (2.15) may be reduced to

$$Br_{XX} + (\mathrm{i}\alpha + \hat\beta + A|r|^2)\,r + \gamma r^* = 0, \qquad (3.1)$$

where

$$\hat\beta \equiv \beta - \tfrac{1}{2}(1-T^2)^2\,\overline{|r|^2}. \qquad (3.2)$$

The solutions of (3.1) are cnoidal waves, including, for $l\uparrow\infty$, the limiting cases of sech and tanh solitary waves. The cnoidal-wave solutions given in Appendix A of M84*b* and by Umeki (1991) are based on (1.1) above, which implicitly assumes $\overline{|r|^2} = 0$ and therefore is invalid for spatially periodic waves. The corrected solution is given by

$$r = \mathrm{cn}\left[\frac{1}{\kappa}\left(\frac{A}{2B}\right)^{1/2} X, \kappa\right] \exp\left\{\tfrac{1}{2}\mathrm{i}\begin{bmatrix}\sin^{-1}(\alpha/\gamma)\\ \pi - \sin^{-1}(\alpha/\gamma)\end{bmatrix}\right\} \quad (0 < \kappa < 1), \qquad (3.3)$$

where

$$\hat\beta \pm (\gamma^2 - \alpha^2)^{1/2} + A(1 - \tfrac{1}{2}\kappa^{-2}) = 0, \quad \overline{|r|^2} = \overline{\mathrm{cn}^2} = \kappa^{-2}\left[\frac{E(\kappa)}{K(\kappa)} - 1 + \kappa^2\right], \quad (3.4\,a,b)$$

the alternatives in (3.3) and (3.4) are vertically ordered, cn is an elliptic cosine of modulus κ (the family parameter), K and E are complete elliptic integrals, and (by assumption) $A > 0$. The corresponding snoidal wave may be obtained, on the assumption that $A < 0$, by replacing cn by sn and A by $-A$ in (3.3) and $\overline{\mathrm{cn}^2}$ by $\overline{\mathrm{sn}^2} = 1 - \overline{\mathrm{cn}^2}$ in (3.4 b). The sech and tanh solutions follow from the cnoidal and snoidal solutions, respectively, through the limit $\kappa \uparrow 1$; note that $\overline{\mathrm{sech}^2} = 0$, whereas $\overline{\mathrm{tanh}^2} = 1$.

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Appendix. Variational calculation

Introducing $q \equiv |r|^2$ in the last term in the Lagrangian density (2.13), we obtain

$$\tfrac{1}{2}\delta \int (q-\bar{q})^2\,\mathrm{d}x = \int (q-\bar{q})\,\mathrm{d}x\left[\delta q - \frac{1}{l}\int \delta q\,\mathrm{d}\hat{x}\right] \tag{A 1a}$$

$$= \int \delta q\,\mathrm{d}x\left[q - \hat{q} - \frac{1}{l}\int (q-\bar{q})\,\mathrm{d}\hat{x}\right] \tag{A 1b}$$

$$= \int (q-\bar{q})\,\delta q\,\mathrm{d}x, \tag{A 1c}$$

where (A 1 b,c) follows from (A 1 a) through the interchange of x and $\hat{x}$, and

$$\frac{1}{l}\int (q-\bar{q})\,\mathrm{d}\hat{x} = \overline{q-\bar{q}} = 0. \tag{A 2}$$

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